Mathematical Physics
(Individual Overall Contest)

Prob. 1 Consider a scalar in a D-dimensional spacetime background, the gravity and the scalar classical dynamics is described by the following total action

$$S = \frac{1}{16\pi G_D}\int d^D x\sqrt{-g}R - \frac{1}{2}\int d^D x\sqrt{-g}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \tag{1}$$

where G_D is the D-dimensional Newton constant, $g = \det g_{\alpha\beta}$ and $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$, i.e., $g_{\alpha\gamma}g^{\gamma\beta} = \delta^\beta_\alpha$.
1) Derive the Einstein equation and the equation of motion for the scalar field, respectively;
2) In any dimension, the Riemann tensor obeys

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}. \tag{2}$$

Now in two dimensions, these relations imply a connection between the Riemann tensor and the Ricci scalar R. Find this precise relation;
3) Using this relation, what is the implication of the Einstein equation obtained in 1) for D = 2 ?

Prob. 2 Consider the four-dimensional U(1) gauge theory: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. One can construct a class of gauge invariant local operators in the following form

$$\mathcal{O}_{2n}(x) \sim \Big(\prod_{n\text{-pairs}}\eta^{\mu_i\mu_j}\Big)(\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_{m-2}}F_{\mu_{m-1}\mu_m})(\partial_{\mu_{m+1}}\ldots\partial_{\mu_{2n-2}}F_{\mu_{2n-1}\mu_{2n}})(x)\,, \tag{3}$$

which contains 2 field strength tensor $F_{\mu\nu}$ and $2n-4$ derivatives ∂_μ. All $2n$ Lorentz indices are contracted in n pairs, thus every operator is a Lorentz scalar. By contracting Lorentz indices in different ways, you may write down different operators for a given n.

Let A_μ be on-shell physical fields in the operator, and solve the following problems:
(1) Derive all the equations that $F_{\mu\nu}$ should satisfy.
(2) In the $n = 2$ case, it should be easy to see that there is only one operator $F_{\mu\nu}F^{\mu\nu}$. Show that, for all possible ways of Lorentz contractions, there is only one independent operator for the $n = 3$ case.
(3) Derive the set of independent operators for the $n = 4$ case.
(4) Derive the set of independent operators for general n.
You may find it convenient to use a short notation for Lorentz indices such that $F_{\mu\nu}F^{\mu\nu} = F_{12}F_{12}$, and $\partial_\rho F^{\mu\nu}\partial^\mu F_{\nu\rho} = \partial_1 F_{23}\partial_2 F_{31}$.